Grade 6 Math Circles

November 8/9/10, 2022

Introduction to Set Theory - Solution

1. Rewrite these set definitions by listing the elements in the set.

   (a) $A = \{x \mid x \text{ is an upper-case letter}\}$.

   (b) $B = \{x \mid x \text{ is a Canadian province or territory}\}$.

   (c) $C = \{x \mid x \text{ is an odd positive integer}\}$.

   (d) $D = \{x \mid x \text{ is a prime number greater than } 2\}$.

   (e) $E = \{x \mid x \text{ is a negative number greater than } 0\}$.

   (f) $F = \{x \mid x \text{ is the empty set}\}$.

   Solution:

   (a) $A = \{\text{A, B, C}, \ldots, \text{ X, Y, Z}\}$.

   (b) $B = \{\text{Alberta, Manitoba, Nunavut}, \ldots, \text{Ontario, Northwest Territories, Yukon}\}$.

   (c) $C = \{1, 3, 5, 7, 9, \ldots\}$.

   (d) $D = \{3, 5, 7, 11, 13, 17, \ldots\}$.

   (e) $E = \varnothing$.

   (f) $F = \{\varnothing\}$.

2. Let $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the universal set.

   Let $A = \{0, 1, 2, 8, 9\}$ and $B = \{1, 3, 5, 7, 9\}$. Find the following sets by listing all the elements.

   (a) $A \cup B$

   (b) $A \cap B$

   (c) A^C

   (d) B^C

   (e) $A \setminus B$

   (f) $B \setminus A$

   (g) $(A \cup B)^C$

Solution:

(a) $A \cup B = \{0, 1, 2, 3, 5, 7, 8, 9\}$

(b) $A \cap B = \{1, 9\}$

(c) $A^C = \{3, 4, 5, 6, 7\}$

(d) $B^C = \{0, 2, 4, 6, 8\}$

(e) $A \setminus B = \{0, 2, 8\}$

(f) $B \setminus A = \{3, 5, 7\}$

(g) $(A \cup B)^C = \{4, 6\}$

3. Let $A = \{x \mid x \text{ is a real number greater than } 0\}$.
Let $B = \{x \mid x \text{ is a real number less than or equal to } 7\}$.
Write the set definitions of the following set by using the elementhood test.

(a) $A \cap B$

(b) $A \cup B$

(c) $B \cap \mathbb{Z}$

(d) $A \cup \mathbb{N}$

Solution:

(a) $A \cap B = \{x \mid x \text{ is a real number greater than 0 and less than or equal to } 7\}$.

We can also write $A \cap B = \{x \in \mathbb{R} \mid 0 < x \leq 7\}$, using mathematical notations.

(b) Since every real number is either in A or B or in both A and B, we have $A \cup B = \{x \mid x \text{ is a real number}\} = \mathbb{R}$.

(c) Notice that $\mathbb{Z} \subseteq \mathbb{R}$. So, $B \cap \mathbb{Z} = \{x \mid x \text{ is an integer less than or equal to } 7\}$.

(d) Notice that $\mathbb{N} \subseteq A$. So, $A \cup \mathbb{N} = A$.

4. Find the cardinality of the following sets

(a) $\{101, 102, 103, 104\}$

(b) $\{\text{dog, cat}\}$

(c) $\{\{a, b\}, b, \{a, b, c\}\}$

(d) $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$

(e) $\{x \in \mathbb{N} \mid x \leq 10\}$ (or $\{x \mid x$ is a natural number less than or equal to $10\}$)

(f) $\{x \mid x \in \mathbb{W} \setminus \mathbb{N}\}$

(g) $\{x \mid x$ is a prime number$\} \cap \{x \mid x$ is an even positive integer$\}$

(h) $\mathbb{Q} \cap \varnothing$

Solution:

(a) 4.

(b) 2.

(c) 3.

(d) 3.

(e) 10.

(f) 1. Since $\{x \mid x \in \mathbb{W} \setminus \mathbb{N}\} = \{0\}$.

(g) 1. Since $\{x \mid x$ is a prime number$\} \cap \{x \mid x$ is an even positive integer$\} = \{2\}$.

(h) 0. Since $\mathbb{Q} \cap \varnothing = \varnothing$.

5. Let A and B represent sets.

(a) Is it always the case that $(A \cap B) \subseteq (A \cup B)$? Why or why not?

(b) Is it always the case that $(A \cup B) \subseteq (A \cap B)$? Why or why not?

(c) If you have $(A \cap B) \subseteq (A \cup B)$ and $(A \cup B) \subseteq (A \cap B)$, what do you know about the two sets A and B?

Solution:

(a) Yes, that is always the case. Since $(A \cup B)$ contains all elements that are in A and all elements that are in B, it must contain all elements that are in both A and B.

(b) No, that is not always the case. Let's suppose that there is an element, x, in A but not in B. Then $x \in (A \cup B)$ but $x \notin (A \cap B)$. Hence, $(A \cup B)$ is not always a subset of $(A \cap B)$.

(c) If we have $(A \cup B) \subseteq (A \cap B)$, then we have:

$$A \subseteq (A \cup B) \subseteq (A \cap B) \subseteq B$$
$$B \subseteq (A \cup B) \subseteq (A \cap B) \subseteq A$$

Thus, we have $A \subseteq B$ and $B \subseteq A$. Therefore, $A = B$.

6. Let A and B represent sets.
   (a) Is it always the case that $(A \setminus B) \subseteq (A \cap B^C)$? Why or why not?
   (b) Is it always the case that $(A \cap B^C) \subseteq (A \setminus B)$? Why or why not?
   (c) What can you conclude from (a) and (b)?

Solution:
(a) Yes, that is always the case. We have

$$\begin{aligned}(A \setminus B) &= (A \cap (A \setminus B)) \\ &\subseteq (A \cap (U \setminus B)) \\ &= (A \cap B^C)\end{aligned}$$

(b) Yes, that is always the case. Let x be an element in the set $(A \cap B^C)$. Then x is both in A and B^C. That is, x is in A but not in B. Thus, x is in $A \setminus B$ as well. This means if there is an element in $A \cap B^C$, then it is also in $A \setminus B$. Therefore, $(A \cap B^C) \subseteq (A \setminus B)$.

(c) We know that $(A \setminus B) = (A \cap B^C)$.

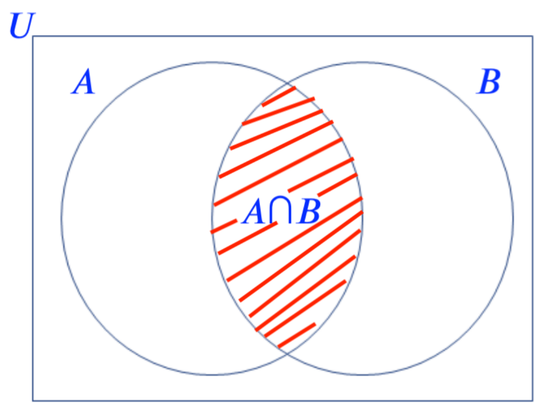
7. Let A and B represent sets.
   (a) Is it always the case that $((A \cup B) \setminus B) \subseteq A$? Why or why not?
   (b) Give an example of two sets A and B for which $(A \cup B) \setminus B \neq A$.

Solution:
(a) Yes, that is always the case. Let x be an element in the set $(A \cup B) \setminus B$. Then x is in $A \cup B$ but not in B. This implies that x must be in A. So, if there is an element in $(A \cup B) \setminus B$, then it is also in A. Therefore $((A \cup B) \setminus B) \subseteq A$.

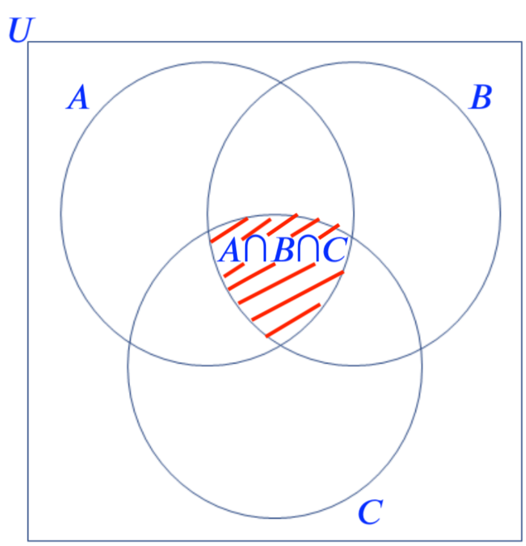
(b) Such example of the sets can be given by $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$.

$$\begin{aligned}(A \cup B) \setminus B &= \{1, 2, 3, 4, 5\} \setminus \{3, 4, 5\} \\ &= \{1, 2\} \\ &\neq A\end{aligned}$$

8. Below is the Venn diagram for 2 sets, A and B. Notice that the overlapping part is the intersection of A and B. Think about what part of the diagram represents the union of A and B.

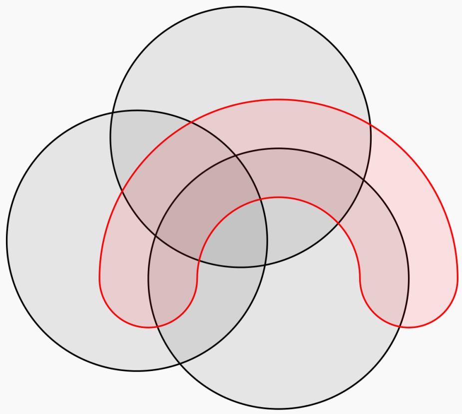


Below is the Venn diagram for 3 sets, A, B, and C. Notice $A \cap B \cap C$ is marked where three circles (sets) are overlapping. Think about what parts of the diagram represent $(A \cap B), (B \cap C), (A \cap C), (A \cup B), (B \cup C), (A \cup C), (A \cup B \cup C)$, etc.



Now, do you think it is possible to draw a Venn diagram for 4 sets, A, B, C, and D? If so, how?

Solution: A possible solution is

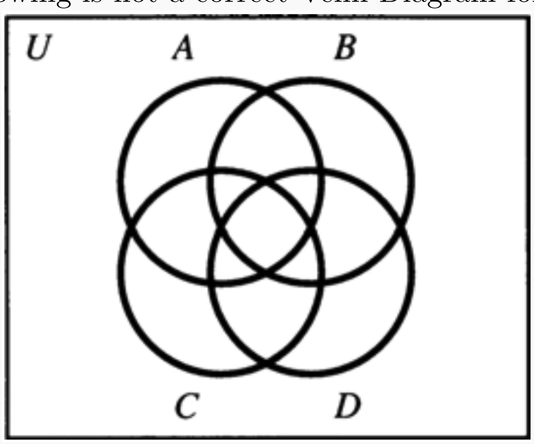


Retrieved from https://en.wikipedia.org/wiki/Venn_diagram

You need to make sure that the diagram has a section to represent each of the following (intersection of) sets:

- 1 set: A, B, C, D
- Intersection of 2 sets: $A \cap B, A \cap C, A \cap D, B \cap C, B \cap D, C \cap D$
- Intersection of 3 sets: $A \cap B \cap C, A \cap B \cap D, A \cap C \cap D, B \cap C \cap D$
- Intersection of 4 sets: $A \cap B \cap C \cap D$

Also, notice that the following is not a correct Venn Diagram for 4 sets.



Retrieved from *How to Prove It* by Velleman

Why is this wrong?

You will not be able to find sections that represent $A \cap D$ and $B \cap C$.

9. **Introduction to Power Set**

Power Set

Given a set A, we define the **power set** of A to be the set

$$\mathbb{P}(A) = \{X \mid X \subseteq A\}.$$

In words, the power set of A, $\mathbb{P}(A)$, is a collection of all subsets of A including $\varnothing$ and A itself.

Let $A = \{0, 1\}, B = \{0, 1, 2\}$, and $C = \{0, 1, 2, 3\}$.

(a) Find $\mathbb{P}(A), \mathbb{P}(B)$, and $\mathbb{P}(C)$.

(b) Find the cardinality of A, B, and C.

(c) Find the cardinality of $\mathbb{P}(A), \mathbb{P}(B)$, and $\mathbb{P}(C)$.

(d) What observation can you make with what you have found in (b) and (c)?

Solution:

(a) We have

$$\begin{aligned}
\mathbb{P}(A) &= \{\varnothing, \{0\}, \{1\}, \{0, 1\}\} \\
\mathbb{P}(B) &= \{\varnothing, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\} \\
\mathbb{P}(C) &= \{\varnothing, \{0\}, \{1\}, \{2\}, \{3\}, \{0, 1\}, \{0, 2\}, \{0, 3\}, \{1, 2\}, \{1, 3\}, \\
&\qquad \{2, 3\}, \{0, 1, 2\}, \{0, 1, 3\}, \{0, 2, 3\}, \{1, 2, 3\}, \{0, 1, 2, 3\}\}
\end{aligned}$$

(b) We have $|A| = 2, |B| = 3$, and $|C| = 4$.

(c) We have $\left|\mathbb{P}(A)\right| = 4, \left|\mathbb{P}(B)\right| = 8$, and $\left|\mathbb{P}(C)\right| = 16$.

(d) Let X be a finite set. When $|X| = n$, then $\left|\mathbb{P}(X)\right| = 2^n$.